

CLASSIFICATION OF THE ABSOLUTE-VALUED ALGEBRAS WITH LEFT-UNIT SATISFYING $x^2(x^2)^2 = (x^2)^2x^2$

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ABSTRACT. We show that every absolute-valued algebra with left-unit satisfying $(x^2, x^2, x^2) = 0$ is finite-dimensional of degree ≤ 4 . Next, we determine such an algebras. In addition to the already known algebras \mathbb{R} , \mathbb{C} , ${}^*\mathbb{C}$, \mathbb{H} , ${}^*\mathbb{H}$, ${}^*\mathbb{H}(i, 1)$, \mathbb{O} , ${}^*\mathbb{O}$, ${}^*\mathbb{O}(i, 1)$ the list is completed by two new algebras not yet specified in the literature.

Summary

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1. INTRODUCTION

A fundamental result of the theory of (real) absolute-valued algebras (AVA) asserts that every finite-dimensional AVA has dimension $n = 1, 2, 4$ or 8 and is isotopic to one of the classical AVA \mathbb{R} , \mathbb{C} , \mathbb{H} or \mathbb{O} [A 47]. On the other hand any familiar identity in arbitrary AVA as associativity [Os 18], commutativity [UW 60], power-associativity ([W 53], [EM 80]) or, even, flexibility [EM 81] carry away finite-dimensionality. However there are infinite-dimensional AVA A in the following two cases:

- (1) A contains a left-unit ([Cu 92], [Rod 92]),
- (2) A satisfies to the identity $(x^2, x^2, x^2) = 0$ [EE 04]. In fact a stronger identity as $(x^2, y, x^2) = 0$ can survive in an infinite-dimensional AVA with the additional property of existence of a non-zero flexible idempotent ([Ch-R 08] Remarks 5.2 p. 850).

The famous paper [Rod 04], in which we find a comprehensive compilation of work on the theory before 2004, appears a list of all AVA of dimension ≤ 2 , namely \mathbb{R} , \mathbb{C} , ${}^*\mathbb{C}$, \mathbb{C}^* , \mathbb{C}^* (see p. 107). These algebras satisfy to the identity $(x^2, x^2, x^2) = 0$, however, only \mathbb{R} , \mathbb{C} and ${}^*\mathbb{C}$ contain a left-unit 1.

A series of studies on four-dimensional AVA has taken place ([St 83], [Ra 99], [CM 05], [F 09]), including a classification through the so-called principal isotopes of \mathbb{H} ($\mathbb{H}(a, b)$, ${}^*\mathbb{H}(a, b)$, $\mathbb{H}^*(a, b)$, $\mathbb{H}^*(a, b)$: a, b being norm-one in \mathbb{H}) ([Ra 99], [CM 05]). The paper [Ra 99] contains a precise description of all four-dimensional AVA with a left-unit ($\mathbb{H}(a, 1)$, ${}^*\mathbb{H}(a, 1)$: a being norm-one in \mathbb{H}) as well as many examples of four-dimensional AVA containing no two-dimensional subalgebras.

A classification for all eight-dimensional AVA with left-unit was given ([Roc 03], [CDD 10]). There are examples of eight-dimensional AVA with left-unit, containing no four-dimensional subalgebras. Such an algebras are characterized, among all eight-dimensional AVA with left-unit, by the triviality of their groups of automorphisms [Roc 03].

A classification of all finite-dimensional AAV has emerged recently [CKMMRR 10]. In this same work a duplication process, characterizing the eight-dimensional AAV which contain four-dimensional subalgebras, has been introduced. This process will plays a decisive role in this present work.

Other recent studies have shown that any left-unit AVA satisfying an identity of the form $(x^p, x^q, x^r) = 0$ (p, q, r being fixed integers in $\{1, 2\}$) is finite-dimensional. Also, for such an algebras it was given:

- (1) a complete classification for $(p, q, r) \neq (2, 2, 2)$,
- (2) a complete classification for $(p, q, r) = (2, 2, 2)$ in dimension ≤ 4 ,
- (3) a partial classification for $(p, q, r) = (2, 2, 2)$ in dimension 8, through \mathbb{O} , ${}^*\mathbb{O}$ and certain algebra ${}^*\mathbb{O}(i, 1)$

([Ch-R 08] Theorem 4.10). Specifically

Theorem. *Every absolute-valued algebra A with left-unit satisfying $(x^p, x^q, x^r) = 0$ for fixed $p, q, r \in \{1, 2\}$ is finite-dimensional. The following table specifies the isomorphisms classes*

A satisfies	The list of isomorphisms classes
$(x, x^q, x^r) = 0$	$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$
$(x^2, x^q, x^r) = 0$ with $(q, r) \neq (2, 2)$	$\mathbb{R}, \mathbb{C}, {}^*\mathbb{C}, \mathbb{H}, {}^*\mathbb{H}, \mathbb{O}, {}^*\mathbb{O}$
$(x^2, x^2, x^2) = 0$	$\mathbb{R}, \mathbb{C}, {}^*\mathbb{C}, \mathbb{H}, {}^*\mathbb{H}, {}^*\mathbb{H}(i, 1)$ if $\dim(A) \leq 4$ contains strictly $\mathbb{O}, {}^*\mathbb{O}, {}^*\mathbb{O}(i, 1)$ if $\dim(A) = 8$

However, the problem of determining all eight-dimensional AVA with left-unit satisfying $(x^2, x^2, x^2) = 0$ remains open. Here we give a complete description of these algebras. We establish the following first basic result (Theorem 2):

Theorem. *Let A be an absolute-valued algebra with left-unit e . Then the following assertions are equivalent:*

- (1) *A satisfies to the identity $(x^2, x^2, x^2) = 0$.*
- (2) *$x^2e = x^2$ for all $x \in A$.*

Under these conditions A is finite-dimensional of degree ≤ 4 . \square

This last result and the use of the duplication process and the form of an eight-dimensional AVA with a left-unit [Roc 03] are key results. They allowed us to have a general expression of an eight-dimensional AVA with left-unit satisfying $(x^2, x^2, x^2) = 0$ through a pair of isometries of Euclidean space \mathbb{H} (Proposition 6). By a laborious calculation, we specify this pair of isometries and reduce the isomorphism classes. The list obtained contains algebras \mathbb{O} , ${}^*\mathbb{O}$, ${}^*\mathbb{O}(i, 1)$ more two new algebras $\tilde{\mathbb{O}}$ and $\tilde{\mathbb{O}}(i)$ not yet specified in the literature.

2. NOTATIONS AND PRELIMINARY RESULTS

By an algebra over a field \mathcal{K} we mean a vector space A over \mathcal{K} endowed with a bilinear mapping $(x, y) \mapsto xy$ from $A \times A$ to A called the product of the algebra.

A non-zero algebra A is said to be a division algebra if for all non-zero $a \in A$ the linear operators $L_a : A \rightarrow A$ $x \mapsto ax$ and $R_a : A \rightarrow A$ $x \mapsto xa$ are bijective.

Let f, g be linear mappings over an algebra A we denote by $A_{f,g}$ the space A with the new product given by the formula $x \odot y = f(x)g(y)$. Also for A we denote by $Aut(A)$ the automorphism group of A , the one of algebra \mathbb{O} is denoted G_2 . An involutive automorphism of A which is different from the identity is said to be a reflection of A .

2.1. The identity $(x^2, x^2, x^2) = 0$. Let A be an algebra over a field \mathcal{K} of characteristic zero. We denote by (x, y, z) the associator $(xy)z - x(yz)$ of $x, y, z \in A$. There are maps $f_n : A \times A \rightarrow A$ with $n = 1, \dots, 5$ such that

$$\begin{aligned} \left((x + \lambda y)^2, (x + \lambda y)^2, (x + \lambda y)^2 \right) &= (x^2, x^2, x^2) + \lambda f_1(x, y) + \dots \\ &\quad + \lambda^5 f_5(x, y) + \lambda^6 (y^2, y^2, y^2) \end{aligned}$$

for all x, y in A and λ in \mathcal{K} . The identity $(x^2, x^2, x^2) = 0$ in A is equivalent to $f_1 = \dots f_5 \equiv 0$. The equality $f_n \equiv 0$ is called the n^{th} identity obtained from $(x^2, x^2, x^2) = 0$ by linearization. The equalities $f_1 \equiv 0$ and $f_2 \equiv 0$ are expressed, respectively, by the following ones:

$$(2.1) \quad (x^2, x^2, xy + yx) + (x^2, xy + yx, x^2) + (xy + yx, x^2, x^2) = 0$$

$$(2.2) \quad \begin{aligned} &(x^2, x^2, y^2) + (x^2, xy + yx, xy + yx) + (x^2, y^2, x^2) \\ &+ (xy + yx, x^2, xy + yx) + (xy + yx, xy + yx, x^2) + (y^2, x^2, x^2) = 0. \end{aligned}$$

We have the following useful preliminary results:

Lemma 1. *Let A be an algebra over a field of characteristic zero satisfying the identity $(x^2, x^2, x^2) = 0$. Assume, in addition, that A contains a left unit e which is not a divisor of zero. Then the equality $(xe)e = x$ holds for all $x \in A$, that is $R_e^2 = I_A$: the identity operator of A .*

Proof. By putting $x = e$ in the equality (2.1) we get:

$$\begin{aligned} 0 &= (e, e, ey + ye) + (e, ey + ye, e) + (ey + ye, e, e) \\ &= (y + ye, e, e) \\ &= \left((y + ye)e - (y + ye) \right) e \\ &= \left((ye)e - y \right) e. \end{aligned}$$

The result is concluded by a simplification to the right by e . \square

Lemma 2. *Let A be an algebra with left unit e such that $x^2e = x^2$ for all $x \in A$. Then the equality $(xe)e = x$ holds for all $x \in A$.*

Proof. Immediate consequence of $(x + e)^2e = (x + e)^2$. \square

2.2. Absolute valued algebra. A nonzero real algebra A is called absolute-valued if it is endowed with a space norm $||\cdot||$ such that $||xy|| = ||x|| ||y||$ for all $x, y \in A$. A finite dimensional absolute valued algebra A is obviously a division algebra and has an underlying Euclidean structure $(A, \langle \cdot | \cdot \rangle)$ with $||x|| = \langle x | x \rangle$ and we have $\langle xy | xz \rangle = \langle x | x \rangle \langle y | z \rangle$, $\langle yx | zx \rangle = \langle y | z \rangle \langle x | x \rangle$ [Cu-R 95].

The degree of a finite-dimensional algebra A is the smallest natural number n such that all single-generated subalgebras of A have dimension $\leq n$. It follows from Albert's paper [A 47] that finite-dimensional absolute-valued algebras are of degree 1, 2, 4 or 8.

For \mathbb{A} equal to either \mathbb{C} , \mathbb{H} or \mathbb{O} , let us denote by ${}^*\mathbb{A}$, \mathbb{A}^* and $\overset{*}{\mathbb{A}}$ the absolute valued algebras obtained by endowing the normed space of \mathbb{A} with the products $x \odot y := \overline{x}y$, $x \odot y := x\overline{y}$ and $x \odot y := \overline{x}\overline{y}$, respectively, where $x \mapsto \overline{x}$ means the standard involution denoted by $\sigma_{\mathbb{A}}$.

Rodríguez gives the list of all absolute-valued algebras of degree two ([Rod 94] Theorem 2.10):

Theorem 1. *The absolute-valued algebras of degree two are \mathbb{C} , ${}^*\mathbb{C}$, \mathbb{C}^* , $\overset{*}{\mathbb{C}}$, \mathbb{H} , ${}^*\mathbb{H}$, \mathbb{H}^* , $\overset{*}{\mathbb{H}}$, \mathbb{O} , ${}^*\mathbb{O}$, \mathbb{O}^* , $\overset{*}{\mathbb{O}}$ and the pseudo-octonion algebra \mathbb{P} .*

Remark 1. *The unique absolute valued algebra of dimension ≤ 2 are equal to either \mathbb{R} , \mathbb{C} , ${}^*\mathbb{C}$, \mathbb{C}^* or $\overset{*}{\mathbb{C}}$ ([Rod 04] p. 107). All these algebras satisfy to $(x^2, x^2, x^2) = 0$ but \mathbb{R} , \mathbb{C} , ${}^*\mathbb{C}$ are the only ones having left unit 1. \square*

It is well known that \mathbb{A} is a quadratic algebra with property $\mathbb{A} = \mathbb{R} \oplus Im(\mathbb{A})$ where the imaginary space

$$Im(\mathbb{A}) = \{x \in \mathbb{A} : x^2 \in \mathbb{R} \text{ and } x \notin \mathbb{R} - \{0\}\}$$

is a vector subspace of \mathbb{A} ([HKR 91] p. 227-228). We denote by $|Re(a)|$ the real part of arbitrary element a in \mathbb{A} .

In [S54, Teorema 1, p. 6], Segre proves the existence of non-zero idempotents in a real or complex finite-dimensional non-zero algebra with no non-zero nilpotents. As a consequence we get the existence of non-zero idempotents in any finite-dimensional absolute-valued algebra A .

Following [R 04, Proposition 1.1, p. 101] or [Cu-R 95, Lemma 2.1, p. 1725] any continuous homomorphism from a normed algebra into an absolute-valued algebra is contractive. In particular, any isomorphism of finite-dimensional absolute-valued algebras is linearly isometric.

Above facts will be used in the sequel without further reference.

2.3. Isometries of Euclidean space \mathbb{H} . A linear isometry of Euclidean space \mathbb{R}^n is said to be proper (resp. improper) if its determinant is positive (resp. negative). We denote by \mathcal{O} , \mathcal{O}^+ , and \mathcal{O}^- , respectively, the orthogonal group of linear isometries of Euclidean space \mathbb{H} , its subgroup of proper linear isometries and its subset of improper linear isometries. Obviously \mathcal{O}^+ , \mathcal{O}^- form a partition of \mathcal{O} .

For any norm-one $a, b \in \mathbb{H}$ the invertible operators $L_a, R_b : \mathbb{H} \rightarrow \mathbb{H}$ are linear isometries. We denote by $T_{a,b}$ the linear isometry $L_a \circ R_b$ given by $T_{a,b}(x) = axb$ for all $x \in \mathbb{H}$.

We consider now the following subsets of \mathcal{O} :

$$\begin{aligned}\mathcal{I}^+ &= \{f \in \mathcal{O}^+ : f \text{ involutive} \}, \\ \mathcal{I}^- &= \{f \in \mathcal{O}^- : f \text{ involutive} \}.\end{aligned}$$

Also if \mathcal{P} belongs in $\{\mathcal{O}^+, \mathcal{O}^-, \mathcal{I}^+, \mathcal{I}^-\}$, we set:

$$\mathcal{P}_1 = \{f \in \mathcal{P} : f(1) = 1\}.$$

We denote by $S(E)$ the unit sphere of every normed space E and we have the following preliminary result:

Lemma 3. *The sets \mathcal{O}^+ , \mathcal{O}^- , \mathcal{O}_1^+ , \mathcal{O}_1^- , \mathcal{I}^+ , \mathcal{I}^- , \mathcal{I}_1^+ , \mathcal{I}_1^- are given by*

- (1) $\mathcal{O}^+ = \{T_{a,b} : a, b \in S(\mathbb{H})\}$.
- (2) $\mathcal{O}^- = \{T_{a,b} \circ \sigma_{\mathbb{H}} : a, b \in S(\mathbb{H})\} := \mathcal{O}^+ \circ \sigma_{\mathbb{H}}$.
- (3) $\mathcal{O}_1^+ = \{T_{a,\bar{a}} : a \in S(\mathbb{H})\}$.
- (4) $\mathcal{O}_1^- = \{T_{a,\bar{a}} \circ \sigma_{\mathbb{H}} : a \in S(\mathbb{H})\} := \mathcal{O}_1^+ \circ \sigma_{\mathbb{H}}$.
- (5) $\mathcal{I}^+ = \{\pm I_{\mathbb{H}}\} \cup \{T_{a,b} : a, b \in S(Im(\mathbb{H}))\}$.
- (6) $\mathcal{I}^- = \{\pm T_{a,a} \circ \sigma_{\mathbb{H}} : a \in S(\mathbb{H})\}$.
- (7) $\mathcal{I}_1^+ = \{I_{\mathbb{H}}\} \cup \{T_{a,\bar{a}} : a \in S(Im(\mathbb{H}))\}$.
- (8) $\mathcal{I}_1^- = \{\sigma_{\mathbb{H}}\} \cup \{T_{a,\bar{a}} \circ \sigma_{\mathbb{H}} : a \in S(Im(\mathbb{H}))\} := \mathcal{I}_1^+ \circ \sigma_{\mathbb{H}}$.

Proof. Assertions (1), (2) are established in ([HKR 91] Theorem (Cayley) p. 215) and assertions (3), (4) are immediate consequences of the previous ones.

Let now norm-one $a, b \in \mathbb{H}$, we have: $T_{a,b}^2 = T_{a^2,b^2}$. On the other hand $T_{a,b} \circ \sigma_{\mathbb{H}} = \sigma_{\mathbb{H}} \circ T_{\bar{b},\bar{a}}$ and we have:

$$\begin{aligned}
 (T_{a,b} \circ \sigma_{\mathbb{H}})^2 &= (T_{a,b} \circ \sigma_{\mathbb{H}}) \circ (\sigma_{\mathbb{H}} \circ T_{\bar{b},\bar{a}}) \\
 &= T_{a,b} \circ T_{\bar{b},\bar{a}} \\
 &= T_{ab,\bar{a}b}.
 \end{aligned}$$

Now

- $T_{a,b}^2 = I_{\mathbb{H}} \Leftrightarrow b^2 = \overline{a^2} \in \{1, -1\}$, that is $a^2 = b^2 = \pm 1$.
- $(T_{a,b} \circ \sigma_{\mathbb{H}})^2 = I_{\mathbb{H}} \Leftrightarrow \bar{a}b = \overline{ab} \in \{1, -1\}$, that is $b = \pm a$.

This shows assertion **(5)** in the first case and assertion **(6)** in the second one by taking into account assertions **(1)**, **(2)**. Assertions **(7)**, **(8)** are deduced from **(5)** and **(6)**. \square

3. ON AVA WITH LEFT UNIT SATISFYING $(x^2, x^2, x^2) = 0$

Rodriguez proved in ([Rod 92] Remark **4.i**) p. 942) and ([Rod 04] Theorem **3.5** p. 133) the following famous result:

Theorem 2. *The norm of every absolute-valued algebra $(A, \|\cdot\|)$ with left-unit e comes from an inner product $\langle \cdot | \cdot \rangle$, and, putting $x^* = 2\langle e|x \rangle e - x$, we have $\langle xy|z \rangle = \langle y|x^*z \rangle$ and $x^*(xy) = \|x\|^2 y$ for all $x, y, z \in A$. \square*

In the rest of this section A will be assumed to be an absolute-valued algebra with left unit and we keep the notations as in Theorem 1.

Remarks 1. .

- (1) *The following equalities, deduced from the ones in Theorem 1, hold for all $x, y, z \in A$ with x orthogonal to e :*

$$(3.2) \quad \langle xy|z \rangle = -\langle y|xz \rangle$$

$$(3.3) \quad x(xy) = -\|x\|^2 y$$

Particularly, the following equalities hold for all $x \in A$ orthogonal to e :

$$(3.4) \quad \langle xe|x \rangle = -\langle e|x^2 \rangle$$

$$(3.5) \quad xx^2 = -\|x\|^2 x$$

Linearizing (3.5), we get, for all $x, y, z \in A$ with x, y orthogonal to e , the following equality:

$$(3.6) \quad x(yz) + y(xz) = -2\langle x|y\rangle z$$

(2) Equality $x^*(xy) = \|x\|^2 y$ gives, for all $x \in A$:

$$(3.7) \quad x(xe) = 2\langle e|x\rangle xe - \|x\|^2 e$$

Linearizing (3.7), we get:

$$(3.8) \quad x(ye) + y(xe) = 2\langle e|x\rangle ye + 2\langle e|y\rangle xe - 2\langle x|y\rangle e, \quad x, y \in A. \square$$

Let $[x, y]$ be the commutator $xy - yx$ of $x, y \in A$. We have the following useful preliminary results:

Lemma 4. Assume that $(xe)e = x$ for all $x \in A$. Then the following equalities hold for all $x \in A$

- (1) $((xe)x)e = x(xe)$.
- (2) $(x(xe))e = (xe)x$.
- (3) $[xe, x] = \langle e|x\rangle[e, x - xe]$.

If, moreover, x is orthogonal to e , then

- (4) $[xe, x] = 0$,
- (5) $(xe)x^2 = 2\langle e|x^2\rangle x + \|x\|^2 xe$,
- (6) $(xe)^2 = 2\langle e|x^2\rangle e - x^2$,
- (7) $x^2 x = -\|x\|^2 xe$.

Proof.

(1) For all $x \in A$, we have:

$$\begin{aligned} (xe)x &= (xe)((xe)e) \\ &= 2\langle e|xe\rangle(xe)e - \|xe\|^2 e \quad \text{by (3.7)} \\ &= 2\langle e|x\rangle\langle e|e\rangle(xe)e - \|x\|^2 e \\ &= 2\langle e|x\rangle x - \|x\|^2 e. \end{aligned}$$

Therefore

$$\begin{aligned} ((xe)x)e &= 2\langle e|x\rangle xe - \|x\|^2 e \\ &= x(xe) \quad \text{by (3.7)}. \end{aligned}$$

(2) We have $(x(xe))e = ((xe.x)e)e = (xe)x$.

(3) Assume first that x is orthogonal to e , and we have

$$\left((xe)x\right)e = -\|x\|^2e.$$

So $(xe)x = -\|x\|^2e = x(xe)$. In general case we have an orthogonal sum $x = (e|x)e + u$. Thus

$$0 = [ue, u] = [xe - \langle e|x \rangle e, x - \langle e|x \rangle e] = [xe, x] + \langle e|x \rangle [e, xe - x].$$

In the sequel of proof x will be assumed to be orthogonal to e .

- (4) Immediate consequence of **(3)**.
 (5) Keeping in mind that xe is orthogonal to e , we put $(y, z) = (xe, x)$ in **(3.6)** and we have

$$x\left((xe)x\right) + (xe)x^2 = -2\langle x|xe \rangle x.$$

The result follows from equalities $(xe)x = x(xe) = -\|x\|^2e$ and $\langle x|xe \rangle = -\langle e|x^2 \rangle$.

- (6) We put $(y, z) = (xe, e)$ in **(3.6)**. The result follows from equality $\langle x|xe \rangle = -\langle e|x^2 \rangle$ and by taking into account that xe is orthogonal to e .
 (7) We just establish the equality $(xe)^2 = 2\langle e|x^2 \rangle e - x^2$ for all $x \in A$ orthogonal to e . So $x^2(xe) = -(xe)^2(xe) + 2\langle e|x^2 \rangle xe$. By comparison with $x^2(xe) = \|x\|^2x + 2\langle e|x^2 \rangle xe$ we get $(xe)^2(xe) = -\|x\|^2x$ for all $x \in A$ orthogonal to e . As xe is also orthogonal to e , and $(xe)e = x$, we have:

$$x^2x = \left((xe)e\right)^2\left((xe)e\right) = -\|xe\|^2xe = -\|x\|^2xe. \square$$

Lemma 5. Assume that $x^2e = x^2$ for all $x \in A$. Then

- (1) The equality $(x^2)^2 = -\|x\|^4e + 2\langle e|x^2 \rangle x^2$ holds for all $x \in A$.
 (2) The equality $x^2(xe) = \|x\|^2x + 2\langle e|x^2 \rangle xe$ holds for all $x \in A$ orthogonal to e .

Proof. .

- (1) Equality **(3.7)** gives $x^2(x^2e) = 2\langle e|x^2 \rangle x^2e - \|x^2\|^2e$ for all $x \in A$. The result is then consequence of hypothesis $x^2e = x^2$ for all $x \in A$.
 (2) Putting $y = x^2$ in **(3.8)** we have

$$xx^2 + x^2(xe) = 2\langle e|x \rangle x^2 + 2\langle e|x^2 \rangle xe - 2\langle x|x^2 \rangle e.$$

Moreover, $\langle x|x^2 \rangle = \langle ex|x^2 \rangle = \|x\|^2\langle e|x \rangle = 0$. The result follows from **(3.5)**. \square

We can now state the following:

Theorem 3. *Let $(A, ||\cdot||, \langle \cdot, \cdot \rangle)$ be an absolute-valued algebra with left-unit e . Then the following assertions are equivalent:*

- (1) *A satisfies to the identity $(x^2, x^2, x^2) = 0$.*
- (2) *$x^2e = x^2$ for all $x \in A$.*

Under these conditions A is finite-dimensional of degree ≤ 4 and we have $(xe)e = x$ for all $x \in A$.

Proof. The implication (2) \Rightarrow (1) is a consequence of the first proposition in Lemma 5. Assume now that A satisfies to $(x^2, x^2, x^2) = 0$. Then e satisfies $(xe)e = x$ for all $x \in A$ by Lemma 1. So $(x + xe, e, x + xe) = 0$ and $(x + xe, x + xe, e) = (x + xe)^2e - (x + xe)^2$ for all $x \in A$. Now, for all $y \in A$, the equality (2.2) gives:

$$\begin{aligned}
0 &= (y + ye, e, y + ye) + (y + ye, y + ye, e) + (y^2, e, e) \\
&= (y + ye)^2e - (y + ye)^2 + y^2 - y^2e \\
&= \left(y^2 + y.ye + ye.y + (ye)^2 \right)e - \left(y^2 + y.ye + ye.y + (ye)^2 \right) + y^2 - y^2e \\
&= \left(y.ye + ye.y + (ye)^2 \right)e - \left(y.ye + ye.y + (ye)^2 \right) \\
&= \left((y.ye)e - ye.y \right) + \left((ye.y)e - y.ye \right) + \left((ye)^2e - (ye)^2 \right) \\
&= (ye)^2e - (ye)^2 \quad \text{by using assertions (1), (2) of Lemma 4.}
\end{aligned}$$

By replacing y by ze and taking into account that $(ze)e = z$, we obtain: $z^2e = z^2$ for all $z \in A$. This shows the implication (1) \Rightarrow (2).

Now, each of assertions (1), (2) leads to $R_e^2 = I_A$. As $L_e = I_A$ the finite dimensionality of algebra A is consequence of ([Rod 04], Theorem 2.2 p. 109). Let now x be nonzero element of A which we can assume, without loss of generality, to be orthogonal to e . By using equalities (3.5), (3.7) and Lemmas 4, 5 we easily verify that the subalgebra $A(x)$ of A generated by x coincide with $\text{Lin}\{e, x, xe, x^2\}$ and we have the following multiplication table:

	e	x	xe	x^2
e	e	x	xe	x^2
x	xe	x^2	$- x ^2e$	$- x ^2x$
xe	x	$- x ^2e$	$2\langle e x^2\rangle e - x^2$	$2\langle e x^2\rangle x + x ^2xe$
x^2	x^2	$- x ^2xe$	$ x ^2x + 2\langle e x^2\rangle xe$	$- x ^4e + 2\langle e x^2\rangle x^2$

Therefore A has degree ≤ 4 . Note that the finite-dimensionality of algebra A can be concluded from here, regardless of ([Rod 04], Theorem 2.2), taking into account the powerful result in [KRR 97]. \square

Now for arbitrary norm-one elements $a, b \in \mathbb{H}$, let $\mathbb{H}(a, b) := \mathbb{H}_1(a, b)$, ${}^*\mathbb{H}(a, b) := \mathbb{H}_2(a, b)$, $\mathbb{H}^*(a, b) := \mathbb{H}_3(a, b)$ and $\mathbb{H}^*(a, b) := \mathbb{H}_4(a, b)$ be the principal isotopes of \mathbb{H} ([R 99] p. 170), that is, the algebras having \mathbb{H} as underlying normed space and products $x \odot y$ given respectively by $axyb$, $\overline{x}ayb$, $axb\overline{y}$, $a\overline{x}\overline{y}b$.

Using these algebras Ramírez constructed exhaustively all 4-dimensional absolute valued algebras and solved the isomorphism problem ([Ra 99] Proposition 2.1 p. 170 and Proposition 2.3 p. 171):

Proposition 1. *Every four-dimensional absolute-valued algebra is isomorphic to a principal isotope of \mathbb{H} . Moreover two principal isotopes $\mathbb{H}_m(a, b)$ and $\mathbb{H}_{m'}(a', b')$ are isomorphic if and only if $m = m'$ and the equalities $a'p = \varepsilon pa$ and $b'p = \delta pb$ hold for some norm-one element $p \in \mathbb{H}$ and some $\varepsilon, \delta \in \{1, -1\}$. \square*

Among these algebras she specifies those having left unit ([Ra 99] Corollary 3.1 p. 172):

Proposition 2. *Let A be a four-dimensional absolute-valued algebra. Then A has left unit if and only if A is isomorphic to $\mathbb{H}(a, 1)$ or ${}^*\mathbb{H}(a, 1)$ for some norm-one element a in \mathbb{H} . Moreover, given norm-one $a, b \in \mathbb{H}$, the algebras $\mathbb{H}(a, 1)$, $\mathbb{H}(b, 1)$ (resp. ${}^*\mathbb{H}(a, 1)$, ${}^*\mathbb{H}(b, 1)$) are isomorphic if and only if $|Re(a)| = |Re(b)|$. \square*

We can now state the following:

Theorem 4. *Let A be an absolute-valued algebra with left of dimension 4 satisfying to $(x^2, x^2, x^2) = 0$. Then A is equal to either \mathbb{H} , ${}^*\mathbb{H}$ or ${}^*\mathbb{H}(i, 1)$. Among these algebras ${}^*\mathbb{H}(i, 1)$ has degree 4 and the other algebras have degree 2.*

Proof. Let a be norm-one in \mathbb{H} . The algebra $\mathbb{H}(a, 1) = (\mathbb{H}, \odot)$ has left unit \overline{a} and we have:

$$\begin{aligned} (x \odot x) \odot \overline{a} &= x \odot x \text{ for all } x \in \mathbb{H} \Leftrightarrow ax^2 = x^2a \text{ for all } x \in \mathbb{H} \\ &\Leftrightarrow a = \pm 1. \end{aligned}$$

On the other hand the algebra ${}^*\mathbb{H}(a, 1) = (\mathbb{H}, *)$ has left unit a and we have:

$$\begin{aligned}
(x * x) \odot a = x * x \text{ for all } x \in \mathbb{H} &\Leftrightarrow xa^2 = a^2x \text{ for all } x \in \mathbb{H} \\
&\Leftrightarrow a^2 = \pm 1.
\end{aligned}$$

By using Theorem 2 we see that $\mathbb{H}(a, 1)$ satisfies to $(x^2, x^2, x^2) = 0$ if and only if $\mathbb{H}(a, 1) = \mathbb{H}(\pm 1, 1)$, which is equal to $\mathbb{H}(1, 1)$ by Proposition 2. But $\mathbb{H}(1, 1) = \mathbb{H}$. Equally ${}^*\mathbb{H}(a, 1)$ satisfies to $(x^2, x^2, x^2) = 0$ if and only if $a^2 = \pm 1$ and we distinguish the following two cases:

- (1) If $a^2 = 1$ then ${}^*\mathbb{H}(a, 1)$ is equal to ${}^*\mathbb{H}$.
- (2) If $a^2 = -1$ then ${}^*\mathbb{H}(a, 1)$ is equal to ${}^*\mathbb{H}(i, 1)$ according to Proposition 1 and the fact that a is conjugated to the complex number i .

The last assertion is an immediate consequence of Theorem 1. \square

Eight-dimensional absolute-valued algebras with a left-unit have been systematically studied in [Roc 03]. As a first basic result, Rochdi proves the following ([Roc 03], Theorem 4.3):

Proposition 3. *The finite-dimensional absolute-valued algebras with a left-unit are precisely those of the form \mathbb{A}_φ , where \mathbb{A} stands for either \mathbb{R} , \mathbb{C} , \mathbb{H} or \mathbb{O} , $\varphi : \mathbb{A} \rightarrow \mathbb{A}$ is a linear isometry fixing 1, and \mathbb{A}_φ denotes the absolute-valued algebra obtained by endowing the normed space of \mathbb{A} with the product $x \odot y := \varphi(x)y$. Moreover, given linear isometries $\varphi, \phi : \mathbb{A} \rightarrow \mathbb{A}$ fixing 1, $\Phi : \mathbb{A}_\varphi \rightarrow \mathbb{A}_\phi$ is an isomorphism of algebras if and only if $\Phi \in G_2$ and $\phi = \Phi \circ \varphi \circ \Phi^{-1}$.*

Also, for \mathbb{A} and φ as in Proposition 3, subalgebras of \mathbb{A}_φ and φ -invariant subalgebras of \mathbb{A} coincide ([Roc 03] Proposition 5.2). Moreover, a linear isometry $\varphi : \mathbb{O} \rightarrow \mathbb{O}$ fixing 1 can be built in such a way that \mathbb{O} has no four-dimensional φ -invariant subalgebra ([Rod 03] Example 3.1). It follows that *there exist eight-dimensional absolute-valued algebras with a left-unit, containing no four-dimensional subalgebras*. Such algebras are characterized, among all eight-dimensional absolute-valued algebras with a left unit, by the triviality of their groups of automorphisms.

4. ALGEBRAS ${}^*\mathbb{O}(a, 1)$

For arbitrary norm-one element $a \in \mathbb{O}$, let ${}^*\mathbb{O}_l(a, 1)$, ${}^*\mathbb{O}_r(a, 1)$ be the algebras having $(\mathbb{O}, \|\cdot\|)$ as underlying normed space and products $x {}_a\odot y$, $x \odot_a y$ given respectively by $(\bar{x}a)y$, $\bar{x}(ay)$. It is easy to see that $({}^*\mathbb{O}_l(a, 1), \|\cdot\|)$, $({}^*\mathbb{O}_r(a, 1), \|\cdot\|)$ are absolute-valued algebras with left unit a . Moreover, if $a^2 = 1$, then the algebras ${}^*\mathbb{O}_l(a, 1)$ and ${}^*\mathbb{O}_r(a, 1)$ are equal to ${}^*\mathbb{O}$ and has degree two.

Any nonzero subalgebra of \mathbb{A} contains 1 [Se 54] and so is invariant under the standard involution of \mathbb{A} . As \mathbb{A} is alternative, Artin's theorem ([Sc 66] Theorem 3.1 p.29) shows that for any $x, y \in \mathbb{A}$, the set $\{x, y, \bar{x}, \bar{y}\}$ is contained in an associative subalgebra of \mathbb{A} . Also this fact will be used in the sequel without further reference.

Proposition 4. *For norm-one a, b in \mathbb{O} the following two assertions are equivalent:*

- (1) $\Phi : \mathbb{O} \rightarrow \mathbb{O}$ is an automorphism such that $\Phi(a) = b$.
- (2) $\Phi : {}^*\mathbb{O}_l(a, 1) \rightarrow {}^*\mathbb{O}_l(b, 1)$ is an isomorphism such that $\Phi(1) = 1$.

Proof. 1. \Rightarrow 2. For all $x, y \in \mathbb{O}$ we have:

$$\begin{aligned} \Phi(x \circ_a y) &= \Phi(\bar{x}a.y) \\ &= \Phi(\bar{x})\Phi(a).\Phi(y) \\ &= \overline{\Phi(x)}b.\Phi(y) \\ &= \Phi(x) \circ_b \Phi(y). \end{aligned}$$

2. \Rightarrow 1. Note that $\Phi(a)$ is the left unit of algebra ${}^*\mathbb{O}_g(b, 1)$ so $\Phi(a) = b$. In the other hand Φ is a linear isometry of Euclidean space \mathbb{H} fixing 1, then commutes with $\sigma_{\mathbb{O}}$. Let now x, y be in \mathbb{O} , we have:

$$(4.9) \quad \Phi(\bar{x}a.y) = \overline{\Phi(x)}\Phi(a).\Phi(y).$$

This gives

$$(4.10) \quad \Phi(ay) = \Phi(a)\Phi(y) \quad \text{and} \quad \Phi(xa) = \Phi(x)\Phi(a).$$

We have:

$$\begin{aligned} \Phi(a)\Phi(xy)\Phi(a) &= \Phi(a.xy.a) \text{ by equalities (4.10)} \\ &= \Phi(ax.ya) \text{ by Middle Moufang identity} \\ &= \Phi\left((\overline{ax} \overline{a}.a)(ya)\right) \\ &= \Phi(\overline{ax} \overline{a})\Phi(a).\Phi(ya) \text{ by equality (4.9)} \\ &= \Phi(ax\overline{a})\Phi(a).\Phi(ya) \\ &= \Phi(a)\Phi(x)\Phi(\overline{a})\Phi(a).\Phi(ya) \text{ by equalities (4.10)} \\ &= \Phi(a)\Phi(x).\Phi(y)\Phi(a) \\ &= \Phi(a).\Phi(x)\Phi(y).\Phi(a) \text{ by Middle Moufang identity} \end{aligned}$$

The result is obtained by simplifying to the right and left by $\Phi(a)$. \square

The group G_2 acts transitively on the sphere $S(Im(\mathbb{O})) := S^6$, that is the mapping $G_2 \rightarrow S^6 \quad \Phi \mapsto \Phi(i)$ is surjective ([Po 85] Lemme 1, p. 269-270). We deduce easily the following result:

Lemma 6. *For every norm-one $a, b \in \mathbb{O}$ the following assertions are equivalent:*

- (1) *There exists $\Phi \in G_2$ such that $\Phi(a) = b$,*
- (2) *$Re(a) = Re(b)$. \square*

Corollary 1. *For every norm-one $a, b \in \mathbb{O}$ the following assertions are equivalent:*

- (1) *${}^*\mathbb{O}_l(a, 1)$ is isomorphic to ${}^*\mathbb{O}_l(b, 1)$,*
- (2) *$|Re(a)| = |Re(b)|$.*

Proof. Consequence of Proposition 1 and Lemma 1. \square

Corollary 2. *Let a be norm-one in $Im(\mathbb{O})$. Then ${}^*\mathbb{O}_l(a, 1)$ is isomorphic to ${}^*\mathbb{O}_l(i, 1)$. \square*

Proposition 5. *Let a be norm-one in \mathbb{O} . The following assertions are equivalent:*

- (1) *${}^*\mathbb{O}_l(a, 1)$ satisfies to the identity $(x^2, x^2, x^2) = 0$,*
- (2) *${}^*\mathbb{O}_r(a, 1)$ satisfies to the identity $(x^2, x^2, x^2) = 0$,*
- (3) *$a^2 = \pm 1$.*

In these conditions the algebras ${}^\mathbb{O}_l(a, 1)$ and ${}^*\mathbb{O}_r(a, 1)$ are isomorphic, we denote them by ${}^*\mathbb{O}(a, 1)$. Moreover, if $a^2 = -1$, then the algebra ${}^*\mathbb{O}(a, 1)$ has degree four.*

Proof. (1) \Leftrightarrow (3). ${}^*\mathbb{O}_l(a, 1)$ has left unit a and for all $x \in \mathbb{O}$, we have: $x {}_a\odot x = \overline{a}ax$, $(x {}_a\odot x) {}_a\odot a = \overline{\overline{a}ax}a^2 = \overline{a} \overline{a}xa^2$. Now

$$\begin{aligned} (x {}_a\odot x) {}_a\odot a = x {}_a\odot x \text{ for all } x \in \mathbb{O} &\Leftrightarrow \overline{a} \overline{a}xa^2 = \overline{a}ax \text{ for all } x \in \mathbb{O} \\ &\Leftrightarrow xa^2 = a^2x \text{ for all } x \in \mathbb{H} \\ &\Leftrightarrow a^2 = \pm 1. \end{aligned}$$

(2) \Leftrightarrow (3). ${}^*\mathbb{O}_r(a, 1)$ has left unit a and for all $x \in \mathbb{O}$, we have: $x {}_a\odot x = \overline{a}ax$, $(x {}_a\odot x) {}_a\odot a = \overline{\overline{a}ax}a^2 = \overline{a} \overline{a}xa^2$. Now

$$\begin{aligned}
 (x \circ_a x) \circ_a a &= x \circ_a x \text{ for all } x \in \mathbb{O} \Leftrightarrow \bar{x} \bar{a} x a^2 = \bar{x} a x \text{ for all } x \in \mathbb{O} \\
 &\Leftrightarrow x a^2 = a^2 x \text{ for all } x \in \mathbb{H} \\
 &\Leftrightarrow a^2 = \pm 1.
 \end{aligned}$$

Assume now that $a^2 = -1$ and consider the mapping $\Phi : {}^*\mathbb{O}_l(a, 1) \rightarrow {}^*\mathbb{O}_r(a, 1)$ $x \mapsto ax\bar{a}$. For all $x, y \in \mathbb{O}$ we have:

$$\begin{aligned}
 \Phi(x \circ_a y) &= a(\bar{x}a.y)\bar{a} \\
 &= (a\bar{x}a)(y\bar{a}) \text{ by Middle Moufang identity} \\
 &= (\overline{ax\bar{a}})(a.y\bar{a}) \\
 &= \Phi(x) \circ_a \Phi(y). \square
 \end{aligned}$$

According to Corollary 2 we can state:

Corollary 3. *Let a be norm-one in $Im(\mathbb{O})$. Then ${}^*\mathbb{O}(a, 1)$ is isomorphic to ${}^*\mathbb{O}(i, 1)$. \square*

5. DUPLICATION PROCESS

Consider the Cayley-Dickson product \bullet in $\mathbb{H} \times \mathbb{H}$. For arbitrary linear isometries f, f', g, g' of \mathbb{H} with $f(1) = f'(1) = 1$ we define on the space $\mathbb{H} \times \mathbb{H}$ the product

$$(x, y) \odot (u, v) = (f(x), g(y)) \bullet (f'(u), g'(v)).$$

We obtain an eight-dimensional absolute-valued algebra $\mathbb{H} \times \mathbb{H}_{(f,g),(f',g')}$, where (f, g) denotes the linear isometry of $\mathbb{H} \times \mathbb{H}$ such that $(f, g)(x, y) = (f(x), g(y))$. Moreover, the subset $\{(x, 0) : x \in \mathbb{H}\}$ of $\mathbb{H} \times \mathbb{H}_{(f,g),(f',g')}$ is a subalgebra isomorphic to $\mathbb{H}_{f,f'}$. For the algebra $\mathbb{H} \times \mathbb{H}_{(f,g),(f',g')}$ the mapping $(x, y) \mapsto (x, -y)$ is a reflection. The algebra $\mathbb{H} \times \mathbb{H}_{(f,g),(f',g')}$ is said to be obtained from algebra $\mathbb{A}_{f,f'}$ by the duplication process [CKMMRR 10]. Among these algebras the ones having left-unit, necessarily equal to $(1, 0)$, are $\mathbb{H} \times \mathbb{H}_{(f,g)} := \mathbb{H} \times \mathbb{H}_{(f,g),(I_{\mathbb{H}}, I_{\mathbb{H}})}$ [Roc 03].

Let A be an eight-dimensional absolute-valued algebra A . It is shown that A contains a four-dimensional subalgebra if and only if is obtained by duplication process ([CKMMRR 10] Theorem 6.4).

Note $\tilde{\mathbb{O}}$, $\tilde{\mathbb{O}}(i)$, respectively, the algebras $\mathbb{H} \times \mathbb{H}_{(\sigma_{\mathbb{H}}, I_{\mathbb{H}})}$, $\mathbb{H} \times \mathbb{H}_{(T_{i,i}\tau\sigma_{\mathbb{H}}, T_{i,i})}$. Clearly, they are absolute-valued algebras with left-unit $(1, 0)$. By Theorem 3 we check easily that they satisfy the identity $(x^2, x^2, x^2) = 0$. In order to show that they provide new examples of eight-dimensional absolute-valued algebras with left-unit satisfying the identity $(x^2, x^2, x^2) = 0$, we proceed as follows:

Let A be an arbitrary algebra with left-unit e , we set

$$A_e = \{x \in A : xe = x\}.$$

Assume now that Φ is an isomorphism from A onto another algebra B . Then B contains a left-unit $\Phi(e)$ and we have: $\Phi(A_e) = A_{\Phi(e)}$. If, moreover, A is finite-dimensional then $\dim(A_{\Phi(e)}) = \dim(A_e)$.

The following table illustrates the subspaces A_e corresponding to the algebras \mathbb{O} , ${}^*\mathbb{O}$, ${}^*\mathbb{O}(i, 1)$, $\tilde{\mathbb{O}}$, $\tilde{\mathbb{O}}(i)$:

Algebra A	Subspace A_e	$\dim(A_e)$
\mathbb{O}	\mathbb{O}	8
${}^*\mathbb{O}$	\mathbb{R}	1
${}^*\mathbb{O}(i, 1)$	$Im(\mathbb{O})$	7
$\tilde{\mathbb{O}}$	$\mathbb{R} \times \mathbb{H}$	5
$\tilde{\mathbb{O}}(i)$	$\mathbb{R}i \times \mathbb{C}^\perp$	3

Corollary 4. *The five algebras \mathbb{O} , ${}^*\mathbb{O}$, ${}^*\mathbb{O}(i, 1)$, $\tilde{\mathbb{O}}$, $\tilde{\mathbb{O}}(i)$ are mutually non-isomorphic. \square*

6. CLASSIFICATION IN DIMENSION 8

Let now A be an eight-dimensional absolute-valued algebra with left-unit containing a four-dimensional subalgebra. Then, according to Proposition 3 and ([CKMMRR 10] Theorem 6.4), A has the form $\mathbb{H} \times \mathbb{H}_{(\varphi, \psi)}$ with left-unit $(1, 0)$, where $\varphi, \psi : \mathbb{H} \rightarrow \mathbb{H}$ are linear isometries such that $\varphi(1) = 1$. We have:

Proposition 6. *The following assertions are equivalent:*

- (1) $\mathbb{H} \times \mathbb{H}_{(\varphi, \psi)}$ satisfies to the identity $(x^2, x^2, x^2) = 0$.
- (2) The pair (φ, ψ) satisfies to following equalities:

$$(6.11) \quad \varphi(\varphi(x)x) = \varphi(x)x \quad \text{for all } x \in \mathbb{H}$$

$$(6.12) \quad \varphi(\bar{x}\psi(x)) = \bar{x}\psi(x) \quad \text{for all } x \in \mathbb{H}$$

$$(6.13) \quad \psi(\psi(y)\bar{x} + y\varphi(x)) = \psi(y)\bar{x} + y\varphi(x) \quad \text{for all } x, y \in \mathbb{H}$$

Proof. Let \odot be the product in algebra $\mathbb{H} \times \mathbb{H}_{(\varphi, \psi)}$. For every $x, y \in \mathbb{H}$, we have:

$$\begin{aligned} (x, y) \odot (x, y) &= (\varphi(x), \psi(y)) \bullet (x, y) \\ &= (\varphi(x)x - \bar{y}\psi(y), \psi(y)\bar{x} + y\varphi(x)) \end{aligned}$$

and

$$\begin{aligned} ((x, y) \odot (x, y)) \odot (1, 0) &= (\varphi(x)x - \bar{y}\psi(y), \psi(y)\bar{x} + y\varphi(x)) \odot (1, 0) \\ &= (\varphi(\varphi(x)x - \bar{y}\psi(y)), \psi(\psi(y)\bar{x} + y\varphi(x))) \bullet (1, 0) \\ &= (\varphi(\varphi(x)x - \bar{y}\psi(y)), \psi(\psi(y)\bar{x} + y\varphi(x))). \end{aligned}$$

Now, the result is concluded by Theorem 3. \square

We have a first precision on the pair (φ, ψ) :

Lemma 7. *The isometries φ, ψ are involutive. So*

$$(\varphi, \psi) \in (\mathcal{I}_1^+ \times \mathcal{I}^+) \cup (\mathcal{I}_1^+ \times \mathcal{I}^-) \cup (\mathcal{I}_1^- \times \mathcal{I}^+) \cup (\mathcal{I}_1^- \times \mathcal{I}^-).$$

Proof. As $\varphi(1) = 1$ and $1^\perp = \text{Im}(\mathbb{H})$ we have $\varphi(\text{Im}(\mathbb{H})) \subseteq \text{Im}(\mathbb{H})$. For arbitrary $x = \alpha + u \in \mathbb{R} \oplus \text{Im}(\mathbb{H}) = \mathbb{H}$, we have $\varphi(x) = \alpha + \varphi(u) \in \mathbb{R} \oplus \text{Im}(\mathbb{H})$ and

$$\begin{aligned} \varphi(x)x &= \alpha^2 + \alpha(u + \varphi(u)) + \varphi(u)u, \\ \varphi(\varphi(x)x) &= \alpha^2 + \alpha(\varphi(u) + \varphi^2(u)) + \varphi(\varphi(u)u). \end{aligned}$$

So

$$\begin{aligned} 0 &= \varphi(\varphi(x)x) - \varphi(x)x \text{ by equality (6.11)} \\ &= \alpha(\varphi^2(u) - u). \end{aligned}$$

As x is arbitrary and $\varphi(1) = 1$ it follows that $\varphi^2 = I_{\mathbb{H}}$. In the other hand the equality **(6.13)** gives $\psi^2 = I_{\mathbb{H}}$, by putting $x = 1$. \square

Remarks 2. *We can easily verify the following properties:*

- (1) $(I_{\mathbb{H}}, \psi)$ satisfies (6.11), (6.12) for all linear isometry ψ of space \mathbb{H} .
- (2) $(I_{\mathbb{H}}, I_{\mathbb{H}})$ satisfies (6.11), (6.12), (6.13).
- (3) $(I_{\mathbb{H}}, -I_{\mathbb{H}})$ does not satisfy (6.13).
- (4) $(\sigma_{\mathbb{H}}, \pm I_{\mathbb{H}})$ satisfies (6.11), (6.12), (6.13).
- (5) $(T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, -I_{\mathbb{H}})$ does not satisfy (6.13) for all norm-one $a \in \mathbb{H} - \mathbb{R}$.
- (6) Let a, b be norm-one in $Im(\mathbb{H})$, $(T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, T_{b,a})$ satisfies (6.12), (6.13). \square

In the following we will give an accurate list for couples (φ, ψ) that satisfy (6.11), (6.12), (6.13).

Lemma 8. φ satisfies (6.11) if and only if $\varphi \in \{I_{\mathbb{H}}\} \cup \mathcal{I}_1^-$. So $I_{\mathbb{H}}$ is the only possible proper linear isometry for φ .

Proof. We check easily that every isometry in $\varphi \in \{I_{\mathbb{H}}\} \cup \mathcal{I}_1^-$ satisfies (6.11). Assume now that φ satisfies (6.11) with $\varphi \in \mathcal{I}_1^+$ and $\varphi \neq I_{\mathbb{H}}$. By Lemma 3 φ can be written $T_{a,\bar{a}}$ for norm-one a in $Im(\mathbb{H})$ and the equality (6.11) gives $(ax)^2 = (xa)^2$ for all $x \in \mathbb{H}$. This leads to an absurdity by $x = a + u$ for norm-one u in $Im(\mathbb{H})$ with u orthogonal to a . The result follows by Lemma 9 and Remark 2 (1). \square

Let $T(x)$ be the trace $x + \bar{x} \in \mathbb{R}$ of an arbitrary element $x \in \mathbb{H}$. It is well known that $T(xy) = T(yx)$ for all $x, y \in \mathbb{H}$ ([HKR 91] p. 207). Note that $[\bar{x}, \bar{y}] = [x, y]$ for all $x, y \in \mathbb{H}$.

We have the following useful result:

Lemma 9. Let a, b be norm-one in \mathbb{H} . Then $(I_{\mathbb{H}}, T_{a,b})$ satisfies (6.13) if and only if $a = b = \pm 1$, or also $T_{a,b} = I_{\mathbb{H}}$.

Proof. Assume that a, b are norm-one in $Im(\mathbb{H})$. Then

$$\begin{aligned}
 (I_{\mathbb{H}}, T_{a,b}) \text{ satisfies (6.13)} &\Leftrightarrow T_{a,b}(T_{a,b}(y)\bar{x} + yx) = T_{a,b}(y)\bar{x} + yx \text{ for all } x, y \in \mathbb{H} \\
 &\Leftrightarrow a(ayb\bar{x} + yx)b = ayb\bar{x} + yx \text{ for all } x, y \in \mathbb{H} \\
 &\Leftrightarrow -yb\bar{x}b + ayxb = ayb\bar{x} + yx \text{ for all } x, y \in \mathbb{H} \\
 &\Leftrightarrow -yb\bar{x} + ayx = ayb\bar{x}\bar{b} + yx\bar{b} \text{ for all } x, y \in \mathbb{H} \\
 &\Leftrightarrow y(x\bar{b} + b\bar{x}) = ay(b\bar{x} + x\bar{b})b \text{ for all } x, y \in \mathbb{H} \\
 &\Leftrightarrow T(b\bar{x})(y - ayb) = 0 \text{ for all } x, y \in \mathbb{H} \\
 &\Leftrightarrow y = ayb \text{ for all } y \in \mathbb{H} \\
 &\Leftrightarrow \bar{a}y = yb \text{ for all } y \in \mathbb{H} \\
 &\Leftrightarrow \bar{a} = b \in \mathbb{R}. \square
 \end{aligned}$$

The following two preliminary results provide a list for couples, solution of **(6.11)**, **(6.12)**, **(6.13)**, of the form $(I_{\mathbb{H}}, \psi)$ and $(\sigma_{\mathbb{H}}, \psi)$:

Lemma 10. $(I_{\mathbb{H}}, \psi)$ satisfies (6.11), (6.12), (6.13) if and only if $\psi \in \mathcal{E}$ where

$$\mathcal{E} = \{I_{\mathbb{H}}\} \cup \{-T_{a,a} \circ \sigma_{\mathbb{H}} : a \in S(\mathbb{H})\}.$$

Proof. The if part of our Lemma follows from Remark 2 (1) and the fact that $T(\bar{x}.\bar{y}a) = T(\bar{y}a.\bar{x})$. Assume now, for the only if part, that $(I_{\mathbb{H}}, \psi)$ satisfies **(6.13)** distinguish the following two cases:

- (1) If $\psi \in \mathcal{I}^+ - \{I_{\mathbb{H}}, -I_{\mathbb{H}}\}$, then ψ can be written $T_{a,b}$ for $a, b \in \mathbb{H}$ with $a^2 = b^2 = -1$. But this can not occur by Lemma 11.
- (2) If $\psi \in \mathcal{I}^-$, then ψ can be written $T_{a,b} \circ \sigma_{\mathbb{H}}$ for norm-one $a, b \in \mathbb{H}$ with $b = \pm a$. The equality **(6.13)** gives

$$(6.14) \quad x\bar{b}y\bar{a}b + \bar{x}\bar{y}b = \bar{y}b\bar{x} + \bar{a}yx$$

We distinguish the following two subcases:

- (1) For $b = -a$ the equality **(6.14)** expresses the fact that $x(\bar{a}y)$ and $(\bar{a}y)x$ have the same trace. So **(6.13)** is true for all $x, y \in \mathbb{H}$ and all ψ in $\{-T_{a,a} \circ \sigma_{\mathbb{H}} : a \in S(\mathbb{H})\}$.
- (2) For $b = a$ we put $y = 1$ in **(6.14)** and we have

$$[x, \bar{a}] = [a, \bar{x}].$$

In the other hand $[a, \bar{x}] = [\bar{a}, x] = -[x, \bar{a}]$. So $[x, \bar{a}] = 0$ for all $x \in \mathbb{H}$ and then $b = a \in \mathbb{R}$. Substituting in **(6.14)** we get $[x, y] = [\bar{y}, \bar{x}]$ for all $x, y \in \mathbb{H}$, absurd.

The result follows by Remarks 2 (2), (3). \square

Lemma 11. $(\sigma_{\mathbb{H}}, \psi)$ satisfies (6.11), (6.12), (6.13) if and only if $\psi = \pm I_{\mathbb{H}}$.

Proof. The if part of our Lemma follows from Remarks 2 (4). Conversely, we show that $(\sigma_{\mathbb{H}}, \psi)$ does not satisfy **(6.11)**, **(6.12)**, **(6.13)** in case $\psi \in (\mathcal{I}^+ - \{\pm I_{\mathbb{H}}\}) \cup \mathcal{I}^-$. We distinguish the following two cases:

- (1) If $\psi \in \mathcal{I}^+ - \{\pm I_{\mathbb{H}}\}$ then ψ can be written $T_{a,b}$ for norm-one $a, b \in \mathbb{H}$ with $a^2 = b^2 = -1$. The equality **(6.12)** gives

$$(6.15) \quad \bar{x}axb \in \mathbb{R} \quad \text{for all } x \in \mathbb{H}$$

By putting $x = 1$ in **(6.15)** we get $b = \pm a$ and therefore $(xa)^2 \in \mathbb{R}$ for all $x \in Im(\mathbb{H})$. But this cannot occur for $x = a + u$ with norm-one u in $Im(\mathbb{H}) \cap a^\perp$.

- (2) If $\psi \in \mathcal{I}^-$ then ψ can be written $\psi = T_{a,b} \circ \sigma_{\mathbb{H}}$ for norm-one $a, b \in \mathbb{H}$ with $b = \pm a$. The equality **(6.12)** gives $(xa)^2 \in \mathbb{R}$ for all $x \in \mathbb{H}$, absurd. \square

Lemma 12. *Let a, ψ be, respectively, norm-one in \mathbb{H} and linear isometry of Euclidian space \mathbb{H} . Then*

- (1) $(T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, \psi)$ satisfies (6.11) if and only if $a^2 = \pm 1$.
- (2) Assume that $a^2 = -1$. Then
 - (a) for norm-one $b, c \in \text{Im}(\mathbb{H})$; $(T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, T_{b,c})$ satisfies (6.12), (6.13) if and only if $c = \pm a$.
 - (b) for norm-one b in \mathbb{H} ; $(T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, T_{b,\pm b} \circ \sigma_{\mathbb{H}})$ does not satisfy (6.12).

Proof.

- (1) Let $\varphi = T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}$. Then

$$\begin{aligned}
 (\varphi, \psi) \text{ satisfies (6.11)} &\Leftrightarrow \varphi(\varphi(x)x) = \varphi(x)x \text{ for all } x \in \mathbb{H} \\
 &\Leftrightarrow \overline{a(\bar{a}x.\bar{a}x)}\bar{a} = a\bar{x}.\bar{a}x \text{ for all } x \in \mathbb{H} \\
 &\Leftrightarrow a(\bar{x}ax\bar{a})\bar{a} = a\bar{x}.\bar{a}x \text{ for all } x \in \mathbb{H} \\
 &\Leftrightarrow ax\bar{a}^2 = \bar{a}x \text{ for all } x \in \mathbb{H} \\
 &\Leftrightarrow x\bar{a}^2 = \bar{a}^2x \text{ for all } x \in \mathbb{H} \\
 &\Leftrightarrow a^2 = \pm 1.
 \end{aligned}$$

- (2) We distinguish the following two cases:

- (a) If $\psi = T_{b,c}$ for norm-one $b, c \in \text{Im}(\mathbb{H})$, then the equality **(6.13)** gives

$$(6.16) \quad by(a\bar{x}.\bar{a} - c\bar{x}.\bar{c})c = y(a\bar{x}.\bar{a} - c\bar{x}.\bar{c}) \quad \text{for all } x, y \in \mathbb{H}$$

We distinguish the following two subcases:

- (i) If $a\bar{x}.\bar{a} - c\bar{x}.\bar{c} = 0$ for all $x \in \mathbb{H}$, then $c = \pm a$.
- (ii) If $a\bar{x}_0.\bar{a} - c\bar{x}_0.\bar{c} := u_0 \neq 0$, we put $y = (a\bar{x}_0.\bar{a} - c\bar{x}_0.\bar{c})^{-1}$ in **(6.16)** and we get $c = -b$. Equality **(6.16)** then gives for $x = x_0$: $byu_0 = yu_0b$ for all $y \in \mathbb{H}$. As right multiplication by u_0 is bijective, we deduce that $bz = zb$ for all $z \in \mathbb{H}$. But this can not occur for norm-one $b \in \text{Im}(\mathbb{H})$.
- (b) If $\psi = T_{b,c} \circ \sigma_{\mathbb{H}}$ with $c = \pm b \in S(\mathbb{H})$, then the equality **(6.12)** gives $a(\bar{b}x)^2 = (\bar{x}b)^2a$ for all $x \in \mathbb{H}$. But this does not occur for $x = ba^{\frac{1}{2}}$ where $a^{\frac{1}{2}}$ is a square root of a .

The result follows from Remarks 2 (5), (6). \square

We summarize the results obtained in this last paragraph as follows:

Proposition 7. *The eight-dimensional absolute-valued algebras with left-unit satisfying $(x^2, x^2, x^2) = 0$ are precisely those of the form $\mathbb{H} \times \mathbb{H}_{(\varphi, \psi)}$, where (φ, ψ) is a pair of linear isometries of Euclidean space \mathbb{H} belonging to the set $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \cup \mathcal{S}_5$ with*

$$\begin{aligned} \mathcal{S}_1 &= \{(I_{\mathbb{H}}, I_{\mathbb{H}})\}, \\ \mathcal{S}_2 &= \{I_{\mathbb{H}}\} \times \{-T_{a,a} \circ \sigma_{\mathbb{H}} : a \in S(\mathbb{H})\}, \\ \mathcal{S}_3 &= \{\sigma_{\mathbb{H}}\} \times \{I_{\mathbb{H}}, -I_{\mathbb{H}}\}, \\ \mathcal{S}_4 &= \{T_{a,\bar{a}} \circ \sigma_{\mathbb{H}} : a^2 = -1\} \times \{I_{\mathbb{H}}\} \\ \mathcal{S}_5 &= \{(T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, T_{b,a}) : a^2 = b^2 = -1\}. \square \end{aligned}$$

We will study now, the isomorphism classes. We begin with the following preliminary results:

Lemma 13. *Let $a \in \mathbb{O}$ such that $a^2 = \pm 1$. Then the mapping $\Phi : {}^*\mathbb{O}(a, 1) \rightarrow \mathbb{O}_{T_{a,\bar{a}} \circ \sigma_{\mathbb{O}}} x \mapsto \bar{a}x$ is an isomorphism of algebras. Therefore $\mathbb{O}_{T_{a,\bar{a}} \circ \sigma_{\mathbb{O}}}$ is isomorphic to ${}^*\mathbb{O}(i, 1)$ when $a^2 = -1$.*

Proof. We note \odot the product ${}^*\mathbb{O}(a, 1)$ and $*$ the one in $\mathbb{O}_{T_{a,\bar{a}} \circ \sigma_{\mathbb{O}}}$. The result is trivial if $a = \pm 1$ and we can assume $a^2 = -1$. For all $x, y \in \mathbb{O}$, we have:

$$\begin{aligned} \Phi(x \odot y) &= \bar{a}(\bar{a}x.y) \\ &= a((\bar{a}x)(a.y)) \\ &= ((a.\bar{a}x)a)(ay) \text{ by Left Moufang identity} \\ &= (a(\overline{\bar{a}x})\bar{a})(\bar{a}y) \\ &= \Phi(x) * \Phi(y). \end{aligned}$$

The proof is concluded by Corollary 2. \square

Lemma 14. *Let a, b, c, d, u be norm-one in \mathbb{H} and let f, g be linear isometries of euclidian space \mathbb{H} commuting with $T_{\bar{u},u}$. Then $(T_{u,\bar{u}}, T_{u,\bar{u}})$ is an automorphism of algebra $\mathbb{H} \times \mathbb{H} = \mathbb{O}$. Moreover $(T_{u,\bar{u}}, T_{u,\bar{u}})$ is an isomorphism from algebra $\mathbb{H} \times \mathbb{H}_{(T_{ua\bar{u}}, ub\bar{u}} \circ f, T_{uc\bar{u}}, ud\bar{u}} \circ g)$ onto algebra $\mathbb{H} \times \mathbb{H}_{(T_{a,b} \circ f, T_{c,d} \circ g)}$.*

Proof.

- (1) The mapping $(T_{u,\bar{u}}, T_{u,\bar{u}}) := \Phi_u : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H} \times \mathbb{H}$ is invertible with inverse $\Phi_u^{-1} = \Phi_{\bar{u}}$. Moreover Φ_u is an automorphism of algebra $\mathbb{H} \times \mathbb{H} = \mathbb{O}$. Indeed, let \bullet be the product in algebra $\mathbb{O} = \mathbb{H} \times \mathbb{H}$ and let $x, y, x', y' \in \mathbb{H}$, we have

$$\begin{aligned}
\Phi_u(x, y) \bullet \Phi_u(x', y') &= (ux\bar{u}, uy\bar{u}) \bullet (ux'\bar{u}, uy'\bar{u}) \\
&= (ux\bar{u}.ux'\bar{u} - \overline{uy'\bar{u}}.uy\bar{u}, uy\bar{u}.\overline{ux'\bar{u}} + uy'\bar{u}.ux\bar{u}) \\
&= (uxx'\bar{u} - \overline{uy'y\bar{u}}, uy\bar{u}.\overline{yx'\bar{u}} + uy'y\bar{u}) \\
&= \Phi_u(xx' - \overline{y'y}, y\bar{u}.\overline{yx'} + y'y) \\
&= \Phi_u((x, y) \bullet (x', y'))
\end{aligned}$$

In other hand

$$\begin{aligned}
\Phi_u \circ (T_{a,b} \circ f, T_{c,d} \circ g) \circ \Phi_u^{-1} &= (T_{u,\bar{u}}, T_{u,\bar{u}}) \circ (T_{a,b} \circ f, T_{c,d} \circ g) \circ (T_{\bar{u},u}, T_{\bar{u},u}) \\
&= (T_{u,\bar{u}} \circ T_{a,b} \circ f \circ T_{\bar{u},u}, T_{u,\bar{u}} \circ T_{c,d} \circ g \circ T_{\bar{u},u}) \\
&= (T_{u,\bar{u}} \circ T_{a,b} \circ T_{\bar{u},u} \circ f, T_{u,\bar{u}} \circ T_{c,d} \circ T_{\bar{u},u} \circ g) \\
&= (T_{ua\bar{u}, ub\bar{u}} \circ f, T_{uc\bar{u}, ud\bar{u}} \circ g).
\end{aligned}$$

The result is concluded from Proposition 3. \square

Lemma 15. *For arbitrary norm-one $a \in \mathbb{H}$ the mapping $\Phi : \mathbb{H} \times \mathbb{H} = \mathbb{O} \rightarrow \mathbb{H} \times \mathbb{H}$ $(x, y) \mapsto (x, ay)$ is an automorphism of algebra \mathbb{O} .*

Proof. Let $(x, y), (x', y')$ be in $\mathbb{H} \times \mathbb{H}$, we have:

$$\begin{aligned}
\Phi(x, y) \bullet \Phi(x', y') &= (x, ay) \bullet (x', ay') \\
&= (xx' - \overline{ay'}ay, ay.\overline{x'} + ay'.x) \\
&= (xx' - \overline{y'}\bar{a}ay, a(y\overline{x'} + y'.x)) \\
&= \Phi(xx' - \overline{y'}y, y\overline{x'} + y'.x) \\
&= \Phi((x, y) \bullet (x', y')). \square
\end{aligned}$$

We get a first result:

Corollary 5. *Let a, b be arbitrary norm-one in $\text{Im}(\mathbb{H})$ and \mathbb{H} , respectively. Then the following three algebras are mutually isomorphic:*

- (1) $\mathbb{H} \times \mathbb{H}_{(T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, I_{\mathbb{H}})}$,
- (2) $\mathbb{H} \times \mathbb{H}_{(I_{\mathbb{H}}, -T_{b,b} \circ \sigma_{\mathbb{H}})}$,
- (3) ${}^*\mathbb{O}(i, 1)$.

Proof. Let \bullet be the product in algebra $\mathbb{O} = \mathbb{H} \times \mathbb{H}$, and let $T_{a,\bar{a}}^{\mathbb{O}}$ be the mapping $\mathbb{O} \rightarrow \mathbb{O}$ $z \mapsto az\bar{a}$. This operator is expressed in terms of pairs (x, y) of quaternion by

$$T_{a,\bar{a}}^{\mathbb{O}}(x, y) = (a, 0) \bullet (x, y) \bullet (\bar{a}, 0).$$

(1) \Rightarrow (3). For arbitrary $(x, y) \in \mathbb{H} \times \mathbb{H}$, we have

$$\begin{aligned} (T_{a,\bar{a}}^{\mathbb{O}} \circ \sigma_{\mathbb{O}})(x, y) &= (a, 0) \bullet \overline{(x, y)} \bullet (\bar{a}, 0) \\ &= (a, 0) \bullet (\bar{x}, -y) \bullet (\bar{a}, 0) \\ &= (a\bar{x}, -ya) \bullet (\bar{a}, 0) \\ &= (a\bar{x}\bar{a}, -ya.a) \\ &= (a\bar{x}\bar{a}, y) \\ &= (T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, I_{\mathbb{H}})(x, y). \end{aligned}$$

So algebras $\mathbb{H} \times \mathbb{H}_{(T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, I_{\mathbb{H}})}$ and $\mathbb{O}_{T_{a,\bar{a}}^{\mathbb{O}} \circ \sigma_{\mathbb{O}}}$ coincide and the mapping

$$\mathbb{H} \times \mathbb{H}_{(T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, I_{\mathbb{H}})} \rightarrow \mathbb{O}_{T_{a,\bar{a}}^{\mathbb{O}} \circ \sigma_{\mathbb{O}}} \quad (x, y) \mapsto (x, y)$$

is an isomorphism of algebras. According to Lemma 13, $\mathbb{H} \times \mathbb{H}_{(T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, I_{\mathbb{H}})}$ is isomorphic to ${}^*\mathbb{O}(i, 1)$.

(2) \Rightarrow (3). Let f be the element $(0, 1) \in \mathbb{H} \times \mathbb{H} = \mathbb{O}$. For arbitrary $(x, y) \in \mathbb{H} \times \mathbb{H}$, we have

$$\begin{aligned} (T_{bf,\overline{bf}}^{\mathbb{O}} \circ \sigma_{\mathbb{O}})(x, y) &= (0, b) \bullet (\bar{x}, -y) \bullet (0, -b) \\ &= -(\bar{y}b, bx) \bullet (0, b) \\ &= -(-\bar{b}bx, b\bar{y}b) \\ &= (x, -b\bar{y}b) \\ &= (I_{\mathbb{H}}, -T_{b,b} \circ \sigma_{\mathbb{H}})(x, y). \end{aligned}$$

So algebras $\mathbb{H} \times \mathbb{H}_{(I_{\mathbb{H}}, -T_{b,b} \circ \sigma_{\mathbb{H}})}$ and $\mathbb{O}_{T_{bf,\overline{bf}}^{\mathbb{O}} \circ \sigma_{\mathbb{O}}}$ coincide and are isomorphic. As bf in norm-one in $Im(\mathbb{O})$, algebra $\mathbb{H} \times \mathbb{H}_{(I_{\mathbb{H}}, -T_{b,b} \circ \sigma_{\mathbb{H}})}$ is isomorphic to ${}^*\mathbb{O}(i, 1)$ by Lemma 13. \square

A second result:

Corollary 6. *Let a, b be arbitrary norm-one in $Im(\mathbb{H})$. Then the algebras $\mathbb{H} \times \mathbb{H}_{(T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, T_{b,a})}$ and $\mathbb{H} \times \mathbb{H}_{(T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, T_{i,i})}$ are isomorphic.*

Proof. There exists norm-one $u \in \mathbb{H}$ such that $ua\bar{u} = i$. Lemma 16 shows that $\mathbb{H} \times \mathbb{H}_{(T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, T_{b,a})}$ is isomorphic to $\mathbb{H} \times \mathbb{H}_{(T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, T_{u\bar{b}u,i})}$.

Let now, b, c be norm-one in $Im(\mathbb{H})$. There exists norm-one $v \in \mathbb{H}$ such that $c = vb\bar{v}$. Now, the mapping

$$(I_H, L_v) := \Phi : \mathbb{H} \times \mathbb{H}_{(T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, T_{b,i})} \rightarrow \mathbb{H} \times \mathbb{H}_{(T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, T_{c,i})} \quad (x, y) \mapsto (x, vy)$$

is an automorphism of algebra $\mathbb{H} \times \mathbb{H} = \mathbb{O}$ by Lemma 15. In the other hand

$$\begin{aligned} \Phi \circ (T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, T_{b,i}) \circ \Phi^{-1} &= (I_{\mathbb{H}}, L_v) \circ (T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, T_{b,i}) \circ (I_{\mathbb{H}}, L_{\bar{v}}) \\ &= (T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, L_v \circ T_{b,i} \circ L_{\bar{v}}) \\ &= (T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, L_v \circ L_b \circ R_i \circ L_{\bar{v}}) \\ &= (T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, L_v \circ L_b \circ L_{\bar{v}} \circ R_i) \\ &= (T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, L_{vb\bar{v}} \circ R_i) \\ &= (T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, L_c \circ R_i) \\ &= (T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, T_{c,i}). \end{aligned}$$

So Φ is an isomorphism from $\mathbb{H} \times \mathbb{H}_{(T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, T_{b,i})}$ onto $\mathbb{H} \times \mathbb{H}_{(T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, T_{c,i})}$ by Proposition 3. Consequently algebra $\mathbb{H} \times \mathbb{H}_{(T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, T_{b,i})}$ is isomorphic to $\mathbb{H} \times \mathbb{H}_{(T_{i,\bar{i}} \circ \sigma_{\mathbb{H}}, T_{i,i})} = \tilde{\mathbb{O}}(i)$. \square

The following table specifies the isomorphism classes

Algebra $\mathbb{H} \times \mathbb{H}_{(\varphi, \psi)}$	Isomorphism classes
$\mathbb{H} \times \mathbb{H}_{(I_{\mathbb{H}}, I_{\mathbb{H}})}$	\mathbb{O}
$\mathbb{H} \times \mathbb{H}_{(I_{\mathbb{H}}, -T_{a,a} \circ \sigma_{\mathbb{H}})} : a \in S(\mathbb{H})$	${}^*\mathbb{O}(i, 1)$
$\mathbb{H} \times \mathbb{H}_{(\sigma_{\mathbb{H}}, I_{\mathbb{H}})}$	$\tilde{\mathbb{O}}$
$\mathbb{H} \times \mathbb{H}_{(\sigma_{\mathbb{H}}, -I_{\mathbb{H}})}$	${}^*\tilde{\mathbb{O}}$
$\mathbb{H} \times \mathbb{H}_{(T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, I_{\mathbb{H}})} : a \in S(Im(\mathbb{H}))$	${}^*\mathbb{O}(i, 1)$
$\mathbb{H} \times \mathbb{H}_{(T_{a,\bar{a}} \circ \sigma_{\mathbb{H}}, T_{b,a})} : a, b \in S(Im(\mathbb{H}))$	$\tilde{\mathbb{O}}(i)$

We can now state the main result:

Theorem 5. *Every absolute-valued algebras A with left-unit satisfying $(x^2, x^2, x^2) = 0$ is finite-dimensional of degree ≤ 4 . Concretely*

- (1) *If $\deg(A) \leq 2$, then A is equal to either $\mathbb{R}, \mathbb{C}, {}^*\mathbb{C}, \mathbb{H}, {}^*\mathbb{H}, \mathbb{O}$ or ${}^*\mathbb{O}$.*
- (2) *If $\deg(A) = 4$, then A is equal to either ${}^*\mathbb{H}(i, 1), {}^*\mathbb{O}(i, 1), \tilde{\mathbb{O}}$ or $\tilde{\mathbb{O}}(i)$.* \square

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